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Efron's Conjecture on Vulnerability to Bias in a Method for Balancing Sequential Trials

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Abstract

Efron (1971) proposed a method for sequential assignment to treatments or control which is in many ways superior to traditional procedures. To analyse the method's susceptibility to accidental bias a criterion concerning the maximum eigenvalue of a fundamental covariance matrix was introduced. On the basis of numerical evidence, Efron conjectured an explicit formula for this eigenvalue. This note gives a proof of that conjecture.

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Disciplines

Biometry | Other Mathematics

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Abstract. Efron (1971) proposed a method for sequential assignment to treatments or control which is in many ways superior to traditional procedures. To analyze the method's susceptibility to accidental bias a criterion concerning the maximum eigenvalue of a fundamental covariance matrix was introduced. On the basis of numerical evidence, Efron conjectured an explicit formula for this eigenvalue. This note gives a proof of Efron's conjecture.

Key Words. Balanced experiment, biased coin design, covariance matrix, maximum eigenvalue, sequential trials.

I. Efron's Biased Coin Design

Suppose subjects are to be assigned sequentially to either treatment or control. If at the time of arrival of a new subject there have been D more subjects assigned to treatment than control, then Efron (1971) suggests the following action:

If $D > 0$, assign treatment with probability q and control with probability p.

If $D = 0$, assign treatment with probability 1/2 and control with probability 1/2.

If $D < 0$, assign treatment with probability p and control with probability q.

This is Efron's <u>biased coin design</u> and will be denoted by BCD(p). This procedure has several benefits over some traditional procedures such as the student sandwich plan, and it has attracted considerable practical and theoretical attention (e.g., Matts and McHugh (1978), Pocock (1979), Pocock and Simon (1975), and Wei (1977, 1978)).

Now suppose that N subjects have been assigned to treatment via a BCD(p) and let T_k be +1 or −1 accordingly as the kth subject is assigned to treatment or control. The vector $\overline{T} = (T_1, T_2, \ldots, T_n)$ has mean $E(\overline{T}) = 0$, and its covariance matrix will be denoted Σ.

After introducing the biased coin designs Efron (1971) argued persuasively (pp. 408-409) that the vulnerability of a balancing design to an accidental bias is sensibly measured by the maximum eigenvalue of the covariance matrix Σ. This in turn was studied by considering the maximum eigenvalue λ_N of the asymptotic covariance of the vector $(T_{h+1}, T_{h+2}, \ldots, T_{h+N})$ as $h \to \infty$. As $N \to \infty$, these λ_N increase to a finite limit λ, and on the basis of considerable numerical evidence, Efron conjectured (p. 411) that for a BCD(p) one has the exact equality $\lambda = 1 + (p-q)^2$. This conjecture is proved in the next section.

II. Proof of Efron's Conjecture

Several important facts from Efron (1971) will be required. We first consider the asymptotic autocovariance function

$$\rho_k \equiv \lim_{n\to\infty} E(T_h\ T_{h+k})$$

and the associated spectral density

$$f(w) \equiv \sum_{-\infty}^{\infty} \rho_k\ e^{-iwk} = 1 + \sum_{k=1}^{\infty} \rho_k \cos(kw) \quad .$$

Here we note that $\Sigma|\rho_k| < \infty$ so $f(w)$ exists as a continuous, even function (Efron (1971), p. 410). Also, we note that the definition of $f(w)$ differs by a factor of 2π from the usual formula for a spectral density.

A key observation of Efron (1971) is that λ can be expressed in terms of $f(w)$.

Lemma 1. The maximum of $f(w)$ equals λ.

The calculation of this maximum then hinges on the following results.

Lemma 2. Setting $r = p/q$, we have $\rho_0 = 1$, $\rho_1 = -\frac{(r-1)^2}{2r(r+1)}$, and $\rho_2 - \rho_1 = \frac{(r-1)^3}{2r(r+1)^2}$.

Lemma 3. For $k \geq 1$, ρ_k is negative and $\rho_{k+1} - \rho_k$ is positive and decreasing.

Lemma 4. $f(\pi) = 1 + \left(\frac{r-1}{r+1}\right)^2 = 1 + (p-q)^2$.

The proof of Lemma 1 is given in Efron (1971), pp. 411-412, and Lemmas 2, 3, and 4 are contained in Efron's Theorem 4. (The only cautions are that $\rho_2 - \rho_1$ is perhaps most easily calculated directly from Efron's Lemmas 5 and 6, p. 416, and that ρ_1 is misprinted as equal to $\rho_2 - \rho_1$ on p. 415.)

The one missing ingredient is supplied by a general result on trigonometric series (Katznelson (1968), p. 22).

Lemma 5. Suppose $\{a_n\}_{n=-\infty}^{\infty}$ is an even sequence of nonnegative numbers which tend to zero as n tends to infinity. If for all $n > 0$

$$a_{n+1} - 2a_n + a_{n-1} \geq 0 \quad ,$$

then the series

$$g(x) = \sum_{n=-\infty}^{\infty} a_n e^{-int}$$

represents a nonnegative, integrable function.

The facts have now been so arranged that it becomes easy to prove Efron's conjecture. Setting $g(x) = f(\pi) - f(x)$ one sees that showing $\max f(x) = f(\pi) = 1 + (p-q)^2$ is equivalent to proving $g(x) \geq 0$. Writing $g(x) = \Sigma_{n=-\infty}^{\infty} a_n e^{-int}$, we have $a_0 = (p-q)^2 = \left(\frac{r-1}{r+1}\right)^2$ and $a_n = -\rho_n$ for $n \neq 0$. By Lemma 3, the a_n are positive; and since $\Sigma|\rho_n| < \infty$, the a_n tend to zero as n tends to infinity. Also, by Lemma 3 we have $a_{n+1} - 2a_n + a_{n-1} \geq 0$ for $n > 1$, since this is the same as $\rho_{k+1} - \rho_k$ decreasing for $k \geq 1$. To apply Lemma 5, it only remains to check $a_2 - 2a_1 + a_0 \geq 0$. But, by Lemma 2,

$$a_2 - 2a_1 + a_0 = - \frac{(r-1)^3}{2r(r+1)^2} - \frac{(r-1)^2}{2r(r+1)} + \left(\frac{r-1}{r+1}\right)^2$$

$$= \frac{(r-1)^2}{r(r+1)^2} \geq 0 \quad ,$$

so all conditions of Lemma 5 are satisfied. This shows $g(x)$ is non-negative and therefore completes the proof of the conjecture.

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